XI. Ideal Fermi Gas

[Focus on 3D (Non-relativistic) Ideal Fermi Gas]

A. Governing Equations

$[g_s = 2 \cdot s + 1 = 2]$ ← ½ (spin-½ for electrons)

$$N = g_s \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \frac{\varepsilon^{1/2}}{e^{(\varepsilon-\mu)/kT}+1}\, d\varepsilon = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \frac{\varepsilon^{1/2}}{e^{(\varepsilon-\mu)/kT}+1}\, d\varepsilon \qquad (1)$$

$$E = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \frac{\varepsilon^{3/2}}{e^{(\varepsilon-\mu)/kT}+1}\, d\varepsilon \qquad (2)$$

$$pV = kT\, \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \varepsilon^{1/2} \ln\left[1 + e^{-(\varepsilon-\mu)/kT}\right] d\varepsilon \qquad (3)$$

An immediate general result from Eq. (3)

$$pV = kT \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \ln\left[1+e^{-(\varepsilon-\mu)/kT}\right] \overbrace{\left(\frac{d}{d\varepsilon}\varepsilon^{3/2}\right)\cdot\frac{2}{3}}^{\varepsilon^{1/2}} d\varepsilon$$

by parts

$$= \cancelto{0}{\left[\begin{matrix}\text{surface}\\ \text{term}\end{matrix}\right]} - \frac{2}{3} kT \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \varepsilon^{3/2} \frac{d}{d\varepsilon} \ln\left[1+e^{-(\varepsilon-\mu)/kT}\right] d\varepsilon$$

$$= -\frac{2}{3} kT \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \left(-\frac{1}{kT}\right) \int_0^\infty \varepsilon^{3/2} \frac{e^{-(\varepsilon-\mu)/kT}}{1+e^{-(\varepsilon-\mu)/kT}} d\varepsilon$$

$$= \frac{2}{3} \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \frac{\varepsilon^{3/2}}{e^{(\varepsilon-\mu)/kT}+1} d\varepsilon$$

$$= \frac{2}{3} E$$

"getting something from nothing"

$$\therefore \boxed{pV = \frac{2}{3}E} \quad (4)$$ for 3D Ideal Fermi Gas

Ex: Does it work for 3D Ideal Bose Gas? Does it work for 3D Classical Ideal Gas?

a temperature implied by $T=0$ Physics

We will look at

$T \ll T_F$

T_F (Fermi-Temperature)

(metals : $T_F \sim 10^4$ K)

0 K

temperature

Zero-Temperature Physics

'dominates Fermi Gas' Physics due to the Pauli Exclusion rule (or Fermi-Dirac distribution at $T = 0$ K)

Low-Temperature Physics

$T \ll T_F$

(usually the case in physical systems)

B. Physical Contexts

Metals : Conduction electrons

$\frac{N}{V}$ = # conduction electrons per unit volume $\equiv n$

n ← conduction electron number density

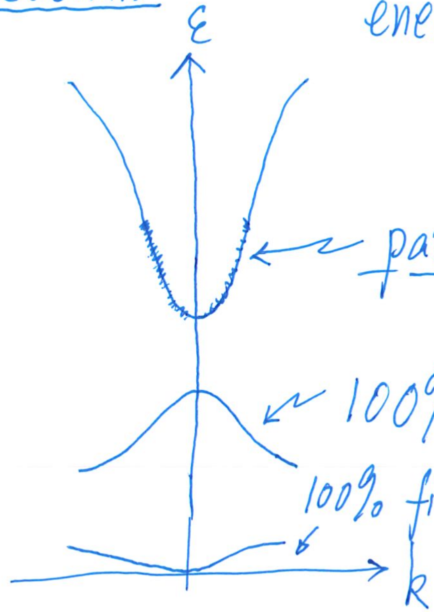
ELEMENT	Z	n (10^{22}/cm^3)
Li (78 K)	1	4.70
Na (5 K)	1	2.65
K (5 K)	1	1.40
Rb (5 K)	1	1.15
Cs (5 K)	1	0.91
Cu	1	8.47
Ag	1	5.86
Au	1	5.90
Be	2	24.7
Mg	2	8.61
Ca	2	4.61
Sr	2	3.55
Ba	2	3.15
Nb	1	5.56
Fe	2	17.0
Mn (α)	2	16.5
Zn	2	13.2
Cd	2	9.27
Hg (78 K)	2	8.65
Al	3	18.1
Ga	3	15.4
In	3	11.5
Tl	3	10.5
Sn	4	14.8
Pb	4	13.2
Bi	5	14.1
Sb	5	16.5

[a] At room temperature (about 300 K) and atmospheric pressure, unless otherwise noted.

$n \sim 10^{22}/\text{cm}^3$
$\sim 10^{28}/\text{m}^3$

Sodium

energy band structure



"conduction electrons"

partially filled by electrons

100% filled

100% filled

from 1s, 2s, 2p atomic orbitals

How do the different $n \equiv \frac{N}{V}$ in different metals affect their properties?

What is the degenerate pressure of a collection of Fermions?

Why is the physics of metals "low-temperature" physics?

Why do the electrons contribute a term linear in T to the heat capacity†?

Why is the physics of $\underbrace{\text{neutron stars}}$ "low-temperature" physics?

mass density $\sim 10^{17}\ \mathrm{Kg/m^3}$ [Earth $\rho \sim 10^3\ \mathrm{Kg/m^3}$]

$\Rightarrow$ neutron number density $\sim 10^{44}/\mathrm{m^3}$

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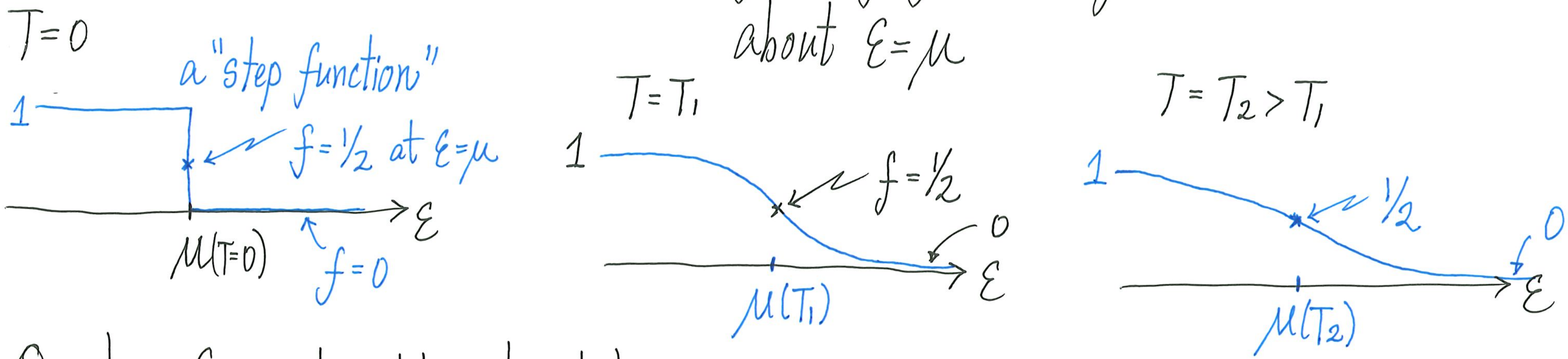
† $C_V \sim \underbrace{\gamma T}_{\text{electrons' contribution}} + \underbrace{bT^3}_{}$ ← vibrations of atoms [Debye model]

C. The Fermi-Dirac Distribution

$$f_{FD}(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/kT}+1} \quad (5)$$

fermion per s.p. state at energy ε

Properties: $f_{FD}(\varepsilon=\mu) = 1/2$ and graph of $f_{FD}(\varepsilon)$ is "symmetric" (artistic sense) about $\varepsilon=\mu$



Question Emerged: How to determine $\mu(T)$?

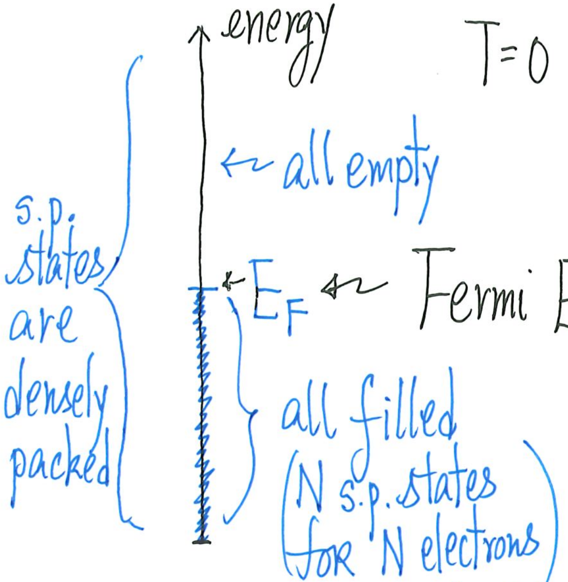
Key Concept

$$N = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty \frac{\varepsilon^{1/2}}{e^{\frac{(\varepsilon-\mu)}{kT}}+1}\,d\varepsilon \quad (6)$$

is an equation that governs $\mu(T)$ for a given $\frac{N}{V}$ (given a material)

D. T=0 Physics : A big part of Fermi Gas Physics is T=0 Physics

$T=0 \Rightarrow$ fill N electrons in from lowest states up until run out of electrons
(this is the ground state of the Fermi Gas)

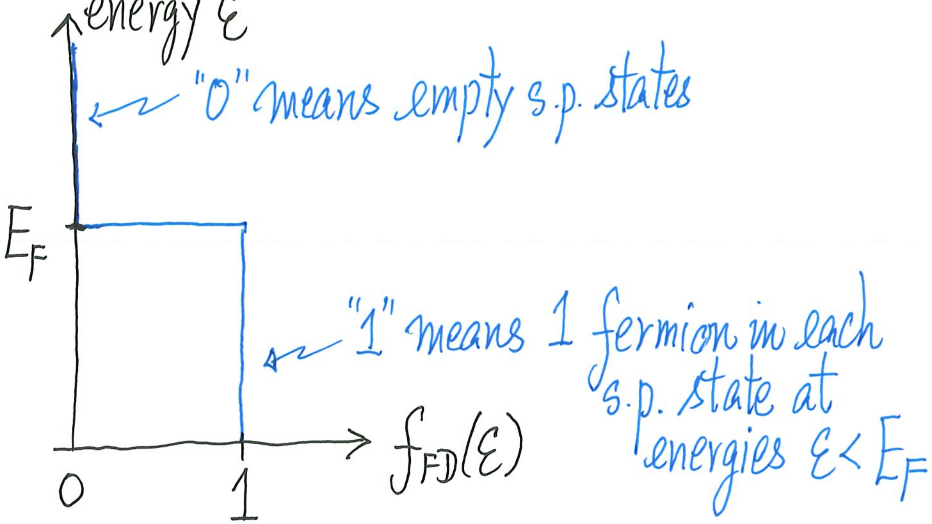


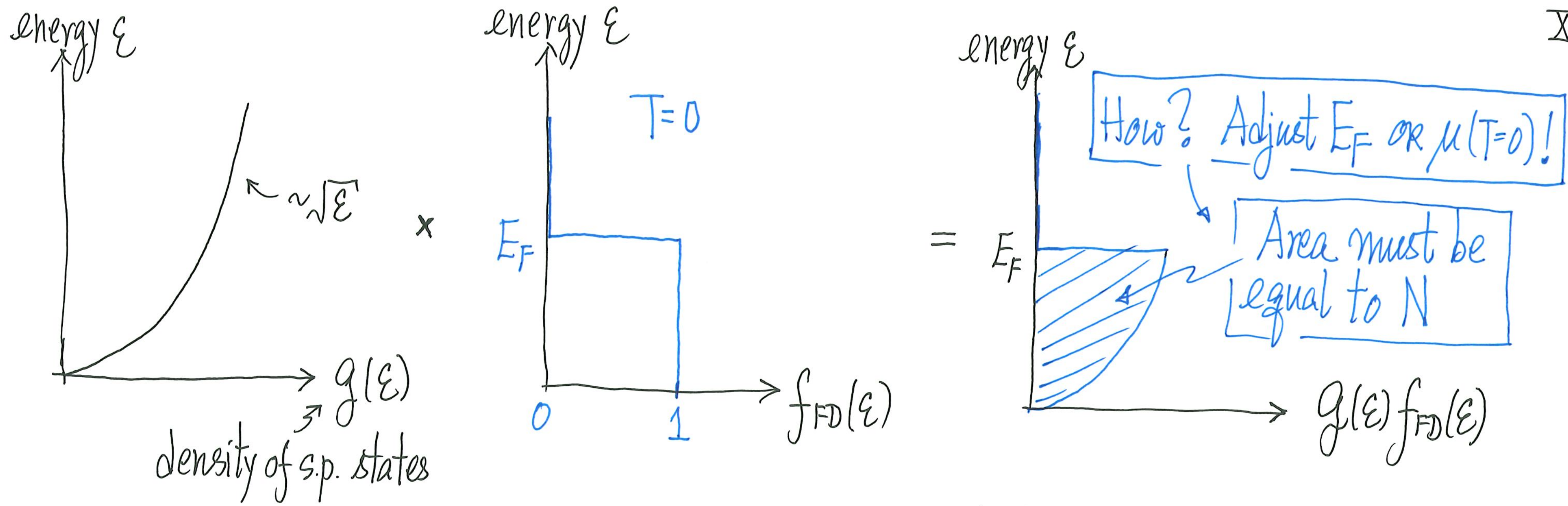
(all s.p. states with $\varepsilon < E_F$ are filled)

Key Concept : $\boxed{\text{Fermi Energy } E_F \text{ is a Zero-temperature concept}}$

This is consistent with $f_{FD}(\varepsilon)$ at $T=0$

$\therefore$ $\boxed{E_F = \mu(T=0) \leftarrow \text{chemical potential at } T=0\,K}$

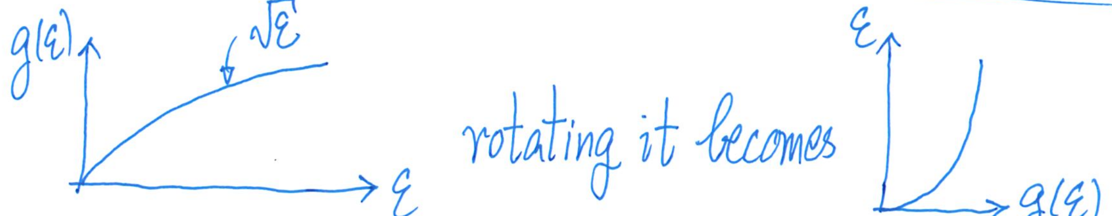




This is the Physical Picture behind $N = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty \frac{\varepsilon^{1/2}}{e^{(\varepsilon-\mu)/kT}+1}\,d\varepsilon$ at $T=0$.

$$N = \int_0^\infty g(\varepsilon) f_{FD}(\varepsilon)\,d\varepsilon \quad \text{(true for any } T\text{, equation fixes } \mu(T)\text{)}$$

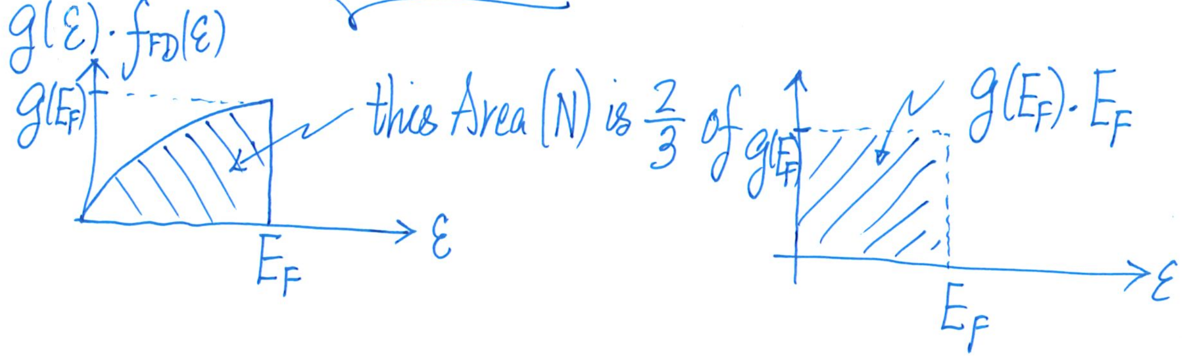
$$= \underbrace{\int_0^{E_F} g(\varepsilon)\,d\varepsilon}_{\text{this is the Area in the figure, see "}E_F\text{" appears in upper limit to adjust for the given } N} \quad (7) \quad (T=0,\ f_{FD}(\varepsilon) \text{ becomes a step function})$$



Writing $g(\varepsilon) = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \varepsilon^{1/2} = A\,\varepsilon^{1/2}$ ← Density of s.p. states at $\varepsilon = E_F$

$$N = A\int_0^{E_F} \varepsilon^{1/2}\, d\varepsilon = A\,\frac{2}{3}E_F^{3/2} = \underbrace{\frac{2}{3}\,(\overbrace{A E_F^{1/2}}^{g(E_F)})\, E_F}$$

$\Rightarrow$ Solve for E_F



$$\therefore\ N = A\,\frac{2}{3}E_F^{3/2} = \frac{2}{3}\,\frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E_F^{3/2}$$

$$\Rightarrow\ \boxed{E_F = \frac{\hbar^2}{2m}\left(3\pi^2\frac{N}{V}\right)^{2/3} = \frac{\hbar^2}{2m}\left(3\pi^2 n\right)^{2/3}} \qquad (8)$$

$\frac{N}{V}$ determines E_F !

$E_F \sim \left(\frac{N}{V}\right)^{2/3} \sim n^{2/3}$ $\Rightarrow$ different metals (different n) have different E_F

$\Rightarrow$ E_F of gold is a property of gold
(not related to the size of sample)

Typical E_F for metals

$$n \sim 10^{22}/\text{cm}^3 , \quad (3\pi^2 n)^{1/3} \sim \frac{1}{\text{length}} \sim \text{order } \text{Å}^{-1}$$

$$E_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3} \sim \text{a few eV} \qquad \left[\frac{\hbar^2}{m} \sim 7.62 \text{ eV}\cdot\text{Å}^2\right]$$

Physics

- Although $T=0$, Pauli Exclusion Rule ($f_{FD}(\varepsilon)$ at $T=0$) forces electrons to fill s.p. states up in energy to E_F
- $E_F \sim$ few eV [which is HIGH ENERGY compared with $\frac{1}{40}$ eV (kT_{room})]

  i.e. $T=0$ Physics sets an Energy Scale of order E_F ($\sim$eV) for metals
- Define $kT_F = E_F$, T_F = Fermi Temperature, $T_F \sim 10^4$ K (the physics is $T=0$)

  i.e. $T=0$ Physics sets a temperature scale of order T_F ($\sim 10^4$K) for metals

[Therefore, 300K metal physics is low-temperature physics!]

FERMI ENERGIES, FERMI TEMPERATURES, FERMI WAVE VECTORS, AND FERMI VELOCITIES FOR REPRESENTATIVE METALS[a]

ELEMENT	E_F	T_F	k_F	v_F
Li	4.74 eV	5.51×10^4 K	$1.12 \times 10^8\ \mathrm{cm}^{-1}$	1.29×10^8 cm/sec
Na	3.24	3.77	0.92	1.07
K	2.12	2.46	0.75	0.86
Rb	1.85	2.15	0.70	0.81
Cs	1.59	1.84	0.65	0.75
Cu	7.00	8.16	1.36	1.57
Ag	5.49	6.38	1.20	1.39
Au	5.53	6.42	1.21	1.40
Be	14.3	16.6	1.94	2.25
Mg	7.08	8.23	1.36	1.58
Ca	4.69	5.44	1.11	1.28
Sr	3.93	4.57	1.02	1.18
Ba	3.64	4.23	0.98	1.13
Nb	5.32	6.18	1.18	1.37
Fe	11.1	13.0	1.71	1.98
Mn	10.9	12.7	1.70	1.96
Zn	9.47	11.0	1.58	1.83
Cd	7.47	8.68	1.40	1.62
Hg	7.13	8.29	1.37	1.58
Al	11.7	13.6	1.75	2.03
Ga	10.4	12.1	1.66	1.92
In	8.63	10.0	1.51	1.74
Tl	8.15	9.46	1.46	1.69
Sn	10.2	11.8	1.64	1.90
Pb	9.47	11.0	1.58	1.83
Bi	9.90	11.5	1.61	1.87
Sb	10.9	12.7	1.70	1.96

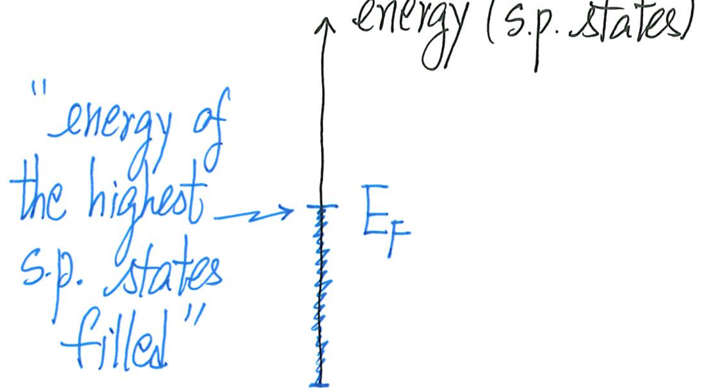
$$v_F = \frac{\hbar k_F}{m}$$

Experimentally, E_F is metal can be measured by X-ray spectroscopy.

[Kick out lower band electrons, and let conduction electrons fall into empty states and emit X-ray in the process.]

[Taken from "Solid State Physics" by Ashcroft and Mermin]

Discussion of $\mu(T=0)$ or E_F



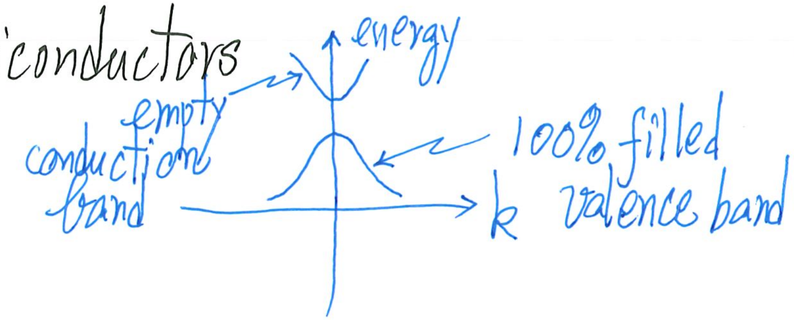
- In metals (Ideal Fermi Gas), E_F or $\mu(T=0)$ is indeed the energy of the highest s.p. states filled
- But, always remember that $\mu(T=0)$ [E_F] or $\mu(T)$ at any temperature is determined by

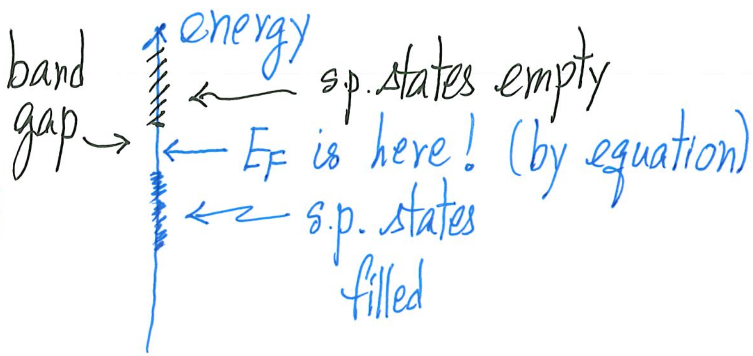
$$N = \int_0^{\infty} g(\varepsilon)\,\frac{1}{e^{(\varepsilon-\mu)/kT}+1}\,d\varepsilon \qquad (1)$$

i.e. tune $\mu(T)$ for given T so that integral gives N

- Why bother with this reminder?

e.g. semiconductors

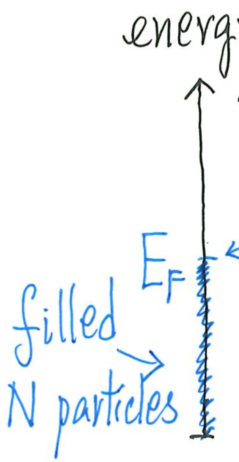




Recall $\mu = \underbrace{\left(\frac{\partial E}{\partial N}\right)_{S,V}}$ (central equation, i.e. 1^{st} + 2^{nd} laws)

slow motion: about adding 1 particle and how energy changes, under the condition of <u>constant entropy</u>

<u>Fermi Gas</u>



Add in the $(N+1)^{th}$ particle

(a) there are s.p. states infinitesimally closed to $\varepsilon = E_F$ for particle to go in

(b) it is still the ground $(T=0)$ state, thus S is kept zero by adding the particle to a state at $\varepsilon = E_F$

(c) $\mu(T=0) = E_F$ makes sense

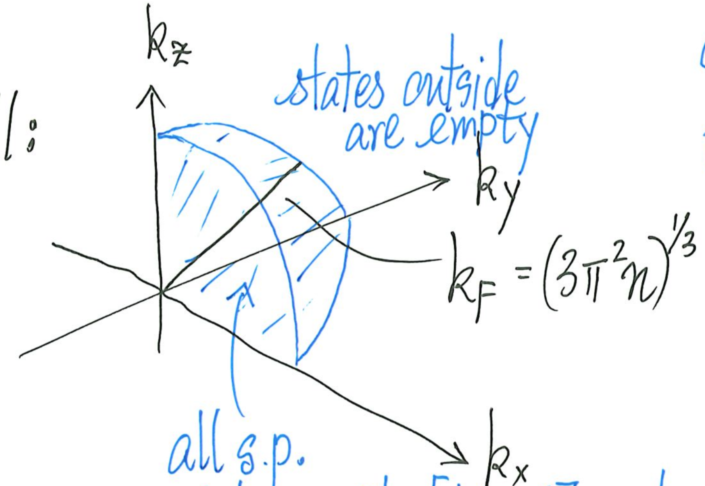
Meaning of $k_F \equiv (3\pi^2 n)^{1/3}$ and the "Fermi Sphere"

$$E_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3} \equiv \frac{\hbar^2 k_F^2}{2m}$$

; $k_F = (3\pi^2 n)^{1/3}$ ~ order $\mathring{A}^{-1}$ for metals

defines a radius in k-space will all s.p. states inside "sphere" being occupied

Recall:



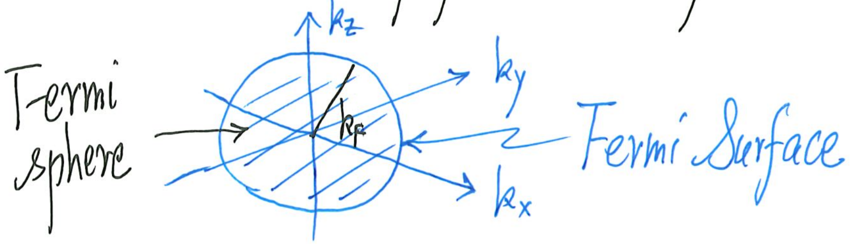
all s.p.
states inside [1/8 of] sphere are occupied

This is the Fermi Sphere

k_F is the Fermi wavevector

The surface [spherical surface of radius k_F] is the Fermi Surface

In solid state physics, usually the whole sphere is used



$$E_F = \mu(T=0) \text{ is fixed by } N = \int_0^\infty g(\varepsilon) f_{FD}(\varepsilon) d\varepsilon$$

- What does $E = \int_0^\infty \varepsilon f_{FD}(\varepsilon) g(\varepsilon) d\varepsilon = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \frac{\varepsilon^{3/2}}{e^{(\varepsilon-\mu)/kT}+1} d\varepsilon$ do?

$\boxed{\text{Plug } \mu(T) \text{ in, the equation gives } E(T)}$ (9)

Energy of Fermi Gas at T=0

From Eq.(2): $E(T=0) = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^{E_F \leftarrow \text{now known}} \varepsilon^{3/2} d\varepsilon = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \frac{2}{5} E_F^{5/2} = \frac{3}{5}\Big[\underbrace{\frac{2}{3}\frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E_F^{3/2}}_{\text{this is } N}\Big] \cdot E_F$

$$\boxed{E(T=0) = \frac{3}{5} N E_F} \quad (10)$$

total energy at T=0 of Fermi Gas

or

$$\boxed{\underbrace{\frac{E(T=0)}{N}}_{\text{energy per particle at } T=0} = \underbrace{\frac{3}{5} E_F}_{60\% \text{ of Fermi Energy}}} \quad (11)$$

(quite high, due to Pauli Exclusion rule)

The third equation gives $pV = \frac{2}{3}E$

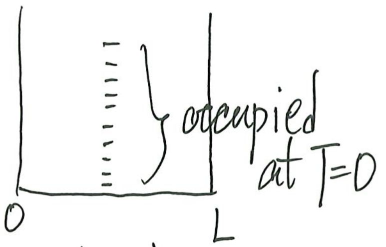
Pressure of Fermi Gas at T = 0 K

pressure at T=0 $\rightsquigarrow$ $P_{T=0} = \frac{2}{3}\left(\frac{E}{V}\right)$ ← "energy density" $= \frac{2}{3}\cdot\frac{3}{5}\frac{N}{V}E_F = \frac{2}{5}\frac{N}{V}\frac{\hbar^2}{2m}\left(3\pi^2\frac{N}{V}\right)^{2/3} \sim \left(\frac{N}{V}\right)^{5/3}$

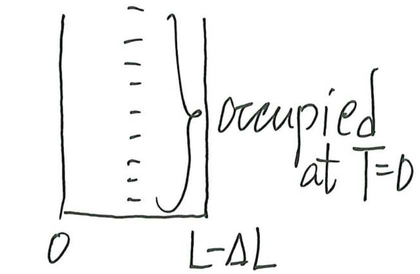
$$P_{T=0} \sim \left(\frac{N}{V}\right)^{5/3} \sim n^{5/3}$$

there is a Pressure even at T=0

$\boxed{\text{this pressure (degenerate pressure) works to oppose gravitation pull (collapse) in dying stars}^{+}}$



wider box



narrower box

[s.p. energies are higher, $\because \sim \frac{n_x^2\pi^2\hbar^2}{2mL^2}$]

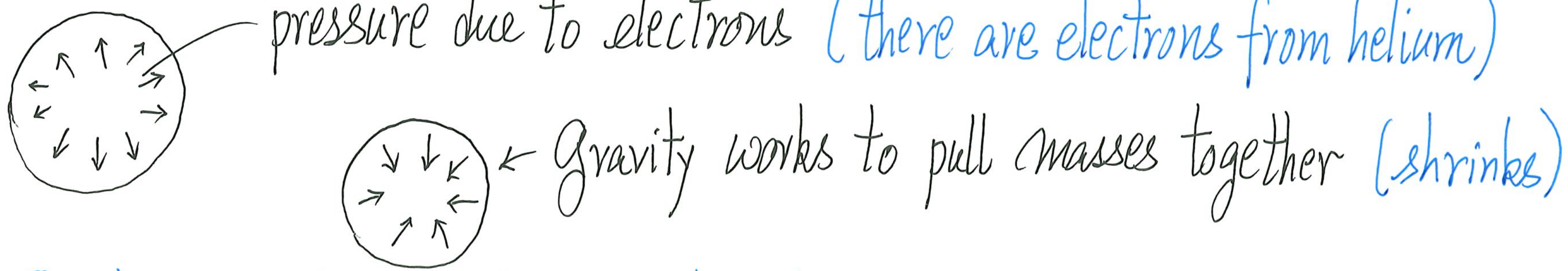
$\therefore$ $-\Delta L$ leads to $+\Delta E$

$\underbrace{\text{(analogous } -\Delta V \text{ leads to } +\Delta U)}_{p = -\frac{\partial U}{\partial V}}$

+ In astrophysics, one may need to consider relativistic particles, which will alter the $g(\varepsilon)$. There is still $P_{T=0}$, but the dependence on n will be different.

Aside: Astrophysical Applications

Would a star (out of fuel) become a white dwarf, or collapse and shrink into a neutron star or a black hole?



Question is whether the quantum (Fermi Gas) pressure is sufficient to oppose the gravitational pull and maintain a stable white dwarf?

Answer: It depends! If Mass of Star $M > M_0 \approx 2.76 \times 10^{30}\,\text{kg} \approx 1.44 M_{sun}$, then gravity will win and star will collapse.

If $M < M_0$, then white dwarf.

M_0 is called the Chandrasekhar Limit (1983 Nobel Physics Prize)

Ex: Should use relativistic version, i.e. $\varepsilon = \sqrt{m^2c^4 + c^2p^2}$ to get $g(\varepsilon)$, it is hard and so try ultra-relativistic case, i.e. $\varepsilon = cp = c\hbar k$ and get $g(\varepsilon)$. Hence, find $P_{T=0}$.

After collapse, conditions allow nuclear process of electron capture $(p + e \to n + \nu_e)$ to occur, and a neutron star is formed.

in nuclei

Neutrons are spin-½ (so fermions). They also provide a Fermi Gas pressure to oppose the gravitational pull ($\rho_{\text{neutron star}} \sim 10^{17}\ \text{Kg/m}^3$) in neutron stars.

Ex: When there are electrons and neutrons, pressure due to electrons is more important. Why?

Summary: You need essential Fermi Gas physics to study Astrophysics.

Summary

- A big part (> 50%) of Fermi Gas Physics is Zero-Temperature Physics
- 3D Ideal Fermi Gas: $E_F = \frac{\hbar^2}{2m}\left(3\pi^2 \frac{N}{V}\right)^{2/3}$, T_F, k_F
- $\mu(T=0) = E_F$ fixed by "N-equation"
- $E_{T=0} = \frac{3}{5} N E_F$ fixed by "E-equation", using E_F fixed by "N-equation"
- $pV = \frac{2}{3} E \Rightarrow P_{T=0} \propto \left(\frac{N}{V}\right)^{5/3}$
- E_F sets a high energy scale ($T=0$ physics), making $T \neq 0$ physics "low-energy physics" (low-temperature physics)

[In contrast, $T=0$ Bose Gas is boring, all particles go into s.p. $\epsilon = 0$ state, $E(T=0) = 0$, $P = 0$. Done!]